HW #1

1) $1, -\frac{1}{3}, \frac{1}{5}, -\frac{1}{7}, \frac{1}{9}, \ldots$

$a_n = \frac{(-1)^n}{(2n+1)}$ starting at $n=0$

2) $4, 10, 28, 82, 244, \ldots$

$a_n = 3^n + 1$ starting at $n=1$

3) $a_n = 1 - (1/3)^n$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 1 - (1/3)^n$$

$$= 1 - \underbrace{\lim_{n\to\infty} (1/3)^n}_{0}$$

$$= 1$$

4) $0, 1, 0, 0, 1, 0, 0, 0, 1, \ldots$

so a_n diverges by oscillating between 0 and 1

5) $a_n = \frac{n^4+4}{n^2+2}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^4+4}{n^2+2}$$

$$= \lim_{n\to\infty} \frac{n^4+4}{n^2+2} \frac{(1/n^4)}{(1/n^4)} = \lim_{n\to\infty} \frac{1+4/n^4}{1/n^2+2/n^4}$$

as $\lim_{n\to\infty} \frac{1}{n^2} + \frac{2}{n^4} = 0$, a_n diverges to ∞

6) $a_n = e^{1/n}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{1/n}$ since e^x is continuous,

$= e^{\lim_{n\to\infty} 1/n}$

$= e^0 = 1$

7) $a_n = \cos^2 n / n$ Note $0 \le \cos^2 n \le 1$,

so $0 \le a_n \le 1/n$

by squeeze theorem, see $\lim_{n\to\infty} 1/n = 0$

$\lim_{n\to\infty} a_n = 0$

9) $a_n = \frac{n^n}{n!} = \left(\frac{n}{1}\right)\cdot\left(\frac{n}{2}\right)\cdot\left(\frac{n}{3}\right)\cdot\ldots\cdot\left(\frac{n}{n}\right)$ each $\frac{n}{\#} \ge 1$

so

$a_n \ge \frac{n}{1}\cdot 1\cdot 1\cdot\ldots\cdot 1 = \frac{n}{1}$

By squeeze theorem, a_n diverges

8) $a_n = n - \sqrt{n}\sqrt{n+1} = \sqrt{n}(\sqrt{n} - \sqrt{n+1})$

$= \sqrt{n}(\sqrt{n} - \sqrt{n+1})\frac{(\sqrt{n} + \sqrt{n+1})}{(\sqrt{n} + \sqrt{n+1})}$ $\frac{-1}{1+\sqrt{x}}$ is continuous

$= \frac{\sqrt{n}(n-(n+1))}{\sqrt{n} + \sqrt{n+1}} = \frac{-\sqrt{n}}{\sqrt{n}+\sqrt{n+1}} = \frac{-1}{\frac{\sqrt{n}}{\sqrt{n}} + \frac{\sqrt{n+1}}{\sqrt{n}}}$

Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{-1}{1 + \frac{\sqrt{n+1}}{\sqrt{n}}} = \lim_{n\to\infty} \frac{-1}{1+\sqrt{\frac{n+1}{n}}} = \frac{-1}{1+\sqrt{\lim_{n\to\infty}\frac{n+1}{n}}} = \frac{-1}{2}$

10) $a_n = \dfrac{5n^2-3n+1}{n^3+1} = \dfrac{5n^2-3n+1}{n^3+1} \dfrac{(1/n^2)}{(1/n^2)}$

$$= \frac{5 - 3/n + 1/n^2}{n + 1/n^2}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\overbrace{5 - 3/n + 1/n^2}^{\to 5}}{\underbrace{n + 1/n^2}_{\to\infty}} = 0$$

there are two!

10) Which values of x make

$$x, \frac{x^2}{2}, \frac{x^3}{3}, \frac{x^4}{4}, \ldots \text{ converge?}$$

$$a_n = \frac{x^n}{n} = \frac{e^{\ln(x^n)}}{n} = \frac{e^{n\ln(x)}}{n} \quad \text{for } x>0$$

If $x>1$, then $\ln(x)=c>0$ so

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{e^{nc}}{n}$$ with $\lim_{n\to\infty} e^{nc}$ and $\lim_{n\to\infty} n$ both diverge to ∞

view as functions $= \lim_{y\to\infty} \dfrac{e^{cy}}{y} = \lim_{y\to\infty} \dfrac{ce^{cy}}{1}$ diverges to ∞

L'Hopital's rule

If $0<x<1$, then $\ln(x)=-d$ with $d>0$, so

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{e^{-nd}}{n} = 0$$ as $\lim_{n\to\infty} e^{-nd} = 0$ and $\lim_{n\to\infty} n$ diverges to ∞.

If $x=1$, $\lim_{n\to\infty} a_n = 1^n = 1$.

10 ctd.) If $-1 \le x < 0$, we can use the squeeze theorem w/ $-\frac{|x|^n}{n} \le a_n \le \frac{|x|^n}{n}$ to see $\lim_{n\to\infty} a_n = 0$

If $x < -1$,

we see $|a_n|$ diverges to ∞, so a_n diverges

11) $\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$

Note $a_n = \sqrt{2a_{n-1}}$,

so $a_n^2 = 2a_{n-1}$, i.e. $a_n^2 - 2a_{n-1} = 0$

Then $0 = \lim_{n\to\infty} a_n^2 - 2a_{n-1}$

The limit $\lim_{n\to\infty} a_n$ exists as $\{a_n\}$ is increasing, bounded by 2.

Say $L = \lim_{n\to\infty} a_n$.

Then $0 = L^2 - 2L$, so $L = 2$ or $L = 0$

since $a_1 > 0$ and $\{a_n\}$ is increasing,

we see $L = 2$.